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Two Mean Value Theorems

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1. INTRODUCTION

The first part of this note deals with a simple proposition, an extension of the first mean value theorem for integrals which does not seem to have been noted before. It provides estimates for integrals of the form

$$I = \int_E f(x)\, g(x)\, dx,$$

where E is a measurable set on the real line, and f and g are real functions on E such that g is nonnegative. If f is continuous, then, as is well known,

$$I = f(\xi) \int_E g\, dx$$

for some $\xi \in E$. Theorem 1 deals with the more general case when f is arbitrary.

In the second part we shall obtain (Theorem 2) an analog of Lagrange's form of the remainder in Taylor's expansion which is valid for analytic functions. Known results of G. Darboux [1] and R. M. McLeod [2] are corollaries, and a theorem of J. M. Robertson [3] is related to a special case of Theorem 2. A description of these three known results will be given below.

2. THE FIRST MEAN VALUE THEOREM FOR INTEGRALS

THEOREM 1. *Let E be a nonempty Lebesgue measurable set on the real line and let f and g be real-valued functions on E such that $g \geqslant 0$ on E. Suppose that the Lebesgue integrals*

$$I = \int_E fg\, dx, \qquad J = \int_E g\, dx$$

exist and have finite values. Then there are numbers x_1, $x_2 \in E$ *such that*

$$f(x_1)\, J \leqslant I \leqslant f(x_2)\, J.$$

Proof. If $J = 0$ then $I = 0$ and we may take as x_1, x_2 any numbers in E. Now let $J > 0$. Then we may assume $J = 1$. Let α and β be the essential lower and upper bounds of f in E, respectively. Then $\alpha \leqslant I \leqslant \beta$.

Case 1. $\alpha = I$. Then $\int (f - \alpha)\, g\, dx = 0$, and hence

$$[f(x) - \alpha]\, g(x) = 0 \qquad \text{p.p. in } E.$$

Since $g > 0$ in a set of positive measure, there is $x_0 \in E$ such that $f(x_0) = \alpha$.

Case 2. $\alpha < I$. Then there is a set of positive measure on which $f(x) < I$. Hence, in either case, there is $x_1 \in E$ such that $f(x_1) \leqslant I$. Applying this to the function $-f$, we find $x_2 \in E$ such that $f(x_2) \geqslant I$, and Theorem 1 follows.

3. Taylor's Theorem

G. Darboux [1] proved the following theorem.

Theorem A. *Let the function* $f(z)$ *be regular at each point of the segment* $[a, b]$, *where* a, b *are distinct complex numbers. Then, for every positive integer* n,

$$f(b) = \sum (0 \leqslant \nu < n) \frac{(b-a)^\nu}{\nu!} f^{(\nu)}(a) + \lambda \frac{(b-a)^n}{n!} f^{(n)}(\zeta)$$

for some $\zeta \in [a, b]$ *and some* λ *such that* $|\lambda| \leqslant 1$.

R. M. McLeod [2] proved

Theorem B. *Let* $f(z)$ *be regular at each point of* $[a, b]$. *Then there are points* ζ_1, $\zeta_2 \in [a, b]$ *and nonnegative numbers* t_1, t_2 *such that* $t_1 + t_2 = 1$ *and*

$$f(b) = f(a) + (b-a)\,[t_1 f'(\zeta_1) + t_2 f'(\zeta_2)].$$

J. M. Robertson [3] proved

Theorem C. *Let* $f(z)$ *be regular at* $z = a$. *Then there is* $\delta > 0$ *such that, given* b *satisfying* $0 < |b - a| < \delta$, *there is* ζ *such that*

$$|\zeta - a| < |b - a| \qquad \textit{and} \qquad f(b) = f(a) + (b-a) f'(\zeta).$$

Theorems A and B are corollaries of Theorem 2 below. Although Theorem C does not seem to follow in the same way, it is mentioned here since it is closely related to the case $n = 1$ of Theorem 2.

Let t', t'' be real numbers, $t' < t''$, and let the complex-valued function $\phi(t)$ be continuous and of bounded variation on $[t', t'']$. The length of the arc

$$C : z = \phi(t); \qquad t' \leqslant t \leqslant t''$$

is

$$\int_{t'}^{t''} | d\phi(t)| .$$

THEOREM 2. *Let a, b be distinct complex numbers and C an arc from a to b of length L. Let the function $f(z)$ be regular at each point of C. Then, for every positive integer n, there are complex numbers t_1, t_2, t_3, ζ_1, ζ_2, ζ_3 such that*

$$\sum t_\kappa = 1; \qquad \sum | t_\kappa | \leqslant \left| \frac{b-a}{L} \right|^n; \qquad \zeta_1 , \zeta_2 , \zeta_3 \in C;$$
$$f(b) = \sum (0 \leqslant \nu < n) \frac{(b-a)^\nu}{\nu!} f^{(\nu)}(a) + \frac{(b-a)^n}{n!} \sum t_\kappa f^{(n)}(\zeta_\kappa).$$

If, in addition, $C = [a, b]$ then t_1, $t_2 \geqslant 0$ and $t_3 = 0$.

To deduce Theorem A from Theorem 2, we choose $\zeta \in [a, b]$ such that $|f^{(n)}(\zeta)| \geqslant |f^{(n)}(z)|$ for all $z \in [a, b]$. Then

$$\left| \sum t_\kappa f^{(n)}(\zeta_\kappa) \right| \leqslant \sum t_\kappa |f^{(n)}(\zeta)| = |f^{(n)}(\zeta)| ,$$

and the assertion follows. Theorem B is the case of Theorem 2 in which $n = 1$ and $C = [a, b]$. The methods used in the proofs of Theorems B and C do not seem to extend to higher derivatives. However, our proof of Theorem 2 has some resemblance with Darboux's proof of his Theorem A.

4. LEMMAS

Let D be a closed bounded subset of the complex plane having at least two elements. In [4] the author introduced the function $N_D(z)$ by means of the definition

$$N_D(z) = \inf \Big\{ | a_1 | + \cdots + | a_m | : m > 0; \sum a_\kappa = 1; \\ \sum a_\kappa z_\kappa = z; z_1 ,\ldots, z_m \in D \Big\} . \tag{1}$$

LEMMA 1. (i) *The function* $N_D(z)$ *is continuous for all complex numbers* z.

(ii) *The infimum in* (1) *is attained with* $m = 3$.

Proof. The assertions follow from Theorem 4 and the corollary to Theorem 15 of [4].

LEMMA 2. *If* D *is a continuous curve in the complex plane then the convex hull of* D *is the union of all segments* $[c, d]$ *such that* $c, d \in D$.

Proof. See [5] and [6].

5. PROOF OF THEOREM 2

Integration by parts, as is well known, shows that

$$\int_C \frac{(b-z)^{n-1}}{(n-1)!} f^{(n)}(z) = f(b) - \sum_0^{n-1} \frac{(b-a)^\nu}{\nu!} f^{(\nu)}(a).$$

Hence

$$f(b) = \sum (0 \leqslant \nu < n) \frac{(b-a)^\nu}{\nu!} f^{(\nu)}(a) + \frac{(b-a)^n}{n!} w,$$

where

$$w = \int_C \frac{n(b-z)^{n-1}}{(b-a)^n} f^{(n)}(z)\, dz.$$

Let C be given by $z = \phi(t)$; $t' \leqslant t \leqslant t''$. Choose a positive integer k and put

$$z_{k\lambda} = \phi\left(t' + \frac{\lambda}{k}(t'' - t')\right) (0 \leqslant \lambda \leqslant k),$$

$$w_k = \sum (0 \leqslant \lambda < k) \frac{n(b - z_{k\lambda})^{n-1}}{(b-a)^n} f^{(n)}(z_{k\lambda}) (z_{k,\lambda+1} - z_{k\lambda}),$$

$$s_k = \sum (0 \leqslant \lambda < k) \frac{n(b - z_{k\lambda})^{n-1}}{(b-a)^n} (z_{k,\lambda+1} - z_{k\lambda}).$$

Then, as $k \to \infty$,

$$s_k \to \int_C \frac{n(b-z)^{n-1}}{(b-a)^n} dz = 1,$$

and there is k_0 such that $s_k \neq 0$ for $k \geqslant k_0$. Let $k \geqslant k_0$ and put

$$\frac{n(b - z_{k\lambda})^{n-1}}{s_k(b-a)^n} (z_{k,\lambda+1} - z_{k\lambda}) = t_{k\lambda} (0 \leqslant \lambda < k).$$

Then

$$\sum (0 \leqslant \lambda < k)\, t_{k\lambda} = 1; \qquad \sum (0 \leqslant \lambda < k)\, t_{k\lambda} f^{(n)}(z_{k\lambda}) = \frac{w_k}{s_k}.$$

Put $\{f^{(n)}(z) : z \in C\} = D$. Then, by definition of $N_D(z)$,

$$N_D\left(\frac{w_k}{s_k}\right) \leqslant \sum | t_{k\lambda} | = \sigma_k ,$$

say. As $k \to \infty$, we have

$$\frac{w_k}{s_k} \to w;$$

$$\sigma_k \to \int_C \left| \frac{n(b-z)^{n-1}}{(b-a)^n}\, dz \right| \leqslant \int_0^L \frac{n(L-s)^{n-1}}{|b-a|^n}\, ds = \left| \frac{L}{b-a} \right|^n.$$

Since, by Lemma 1 (i), $N_D(z)$ is a continuous function of z, we conclude that

$$N_D(w) \leqslant \left| \frac{L}{b-a} \right|^n,$$

and Lemma 1(ii) completes the proof of the theorem, except for the last part. If $C = [a, b]$, then $N_D(w) \leqslant 1$, which means that w lies in the convex hull of D, and Lemma 2 proves the last part of Theorem 2.

References

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